

Multivariate planar curves: definition, alignment and statistical analysis

Cédric Beaulac

Université du Québec à Montréal

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Collaborators involved:



**Issam-Ali
Moindjié**
Perpignan



**Marie-Hélène
Descary**
UQAM



**Mélanie
Raymond**
UQAM



**Louis-Pierre
Ménard**
UQAM

Multivariate Planar Curves: Definition, Alignment & Statistical Analysis

Image data

Edge and contour detection

Functional data analysis

Functional contour alignment

Shape Analysis

Extension to multivariate planar curves

Image data

Image Data

Images have been natively captured and stored in matrix format since cameras became digital.

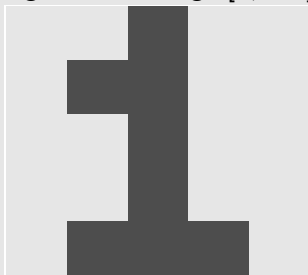
The element (i, j) represents the color intensity at pixel $[i, j]$.

For black-and-white or grayscale images, the color intensity is an integer in the range $[0, 255]$.

A color image is typically represented using three matrices with integer entries in $[0, 255]$.

Image Data

For black-and-white or grayscale images, the pixel intensity is an integer in the range $[0, 255]$.



Black-and-white image of the digit '1'

$$\begin{bmatrix} 255 & 255 & 0 & 255 & 255 \\ 255 & 0 & 0 & 255 & 255 \\ 255 & 255 & 0 & 255 & 255 \\ 255 & 255 & 0 & 255 & 255 \\ 255 & 0 & 0 & 0 & 255 \end{bmatrix}$$

Matrix representation (pixel intensities)

Image Data Analysis

Typical approaches for image analysis are designed to analyze matrices.

Filtering and convolution are matrix operations that perform linear combinations of neighboring pixels.

Powerful predictive models can then be built by learning convolution weights within broader machine learning frameworks.

Recently, there has been interest in developing **transformer models** for image analysis.

Image Data Analysis: pixels

The pixel representations still present several challenges:

- ▶ Difficult interpretation,
- ▶ Large data objects (high-resolution images and videos),
- ▶ Generalization issues (sensitivity to resolution and acquisition technology).

Our Solution

We want to stop looking at images as simple collections of pixels.

Instead, images can be analyzed as collections of objects defined by their shapes, textures, and colors.

Over the last few years, we developed a framework for the analysis of shapes extracted from images.

Edge and contour detection

Edge and contour detection

The first step to analyse shapes in images is to extract them.

For this, we use contour detection techniques.

Edge and contour detection can provide us with flood fill images (or masks).

This is not an easy task.

Edge and contour detection

Assume we have a mask obtained through edge detection.



Photo of a bat.



Flood fill image.

Planar Closed Curves

We extract a functional representation for the contour (a planar closed curve), represented through coordinates:

$$C(t) = (X(t), Y(t)),$$

where $t \in [0, 1]$ represents the proportion of the curve traveled from the start ($t = 0$) to the end ($t = 1$).

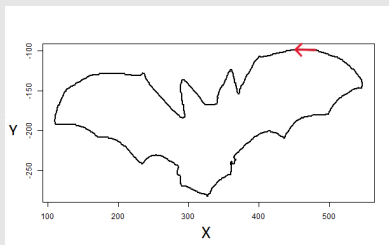
For closed curves, $C(0) = C(1)$.

Traveling along the contour

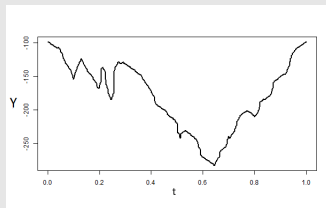
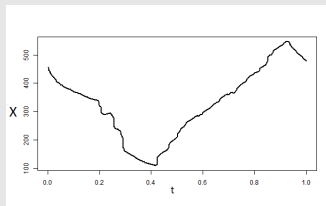
The marching square algorithm (Mantz et al. 2008) will provide us with an ordered sequence of pixels:

$$[(x[1], y[1]), (x[2], y[2]), \dots, (x[T], y[T])] \quad (1)$$

Traveling along the contour



Contour of the bat.



Functional Data Analysis

Functional Data

Functional data analysis (FDA) is a field of statistics concerned with studying data assumed to be realizations of smooth functions,

$$x(t),$$

or higher-dimensional objects such as surfaces,

$$x(s, t).$$

The domain may represent time, space, or more general multidimensional structures.

Basis Expansions

A standard approach consists of representing functions using basis expansions,

$$\tilde{x}(t) = \sum_{k=1}^K c_k B_k(t),$$

where $\{B_k(t)\}$ is a collection of basis functions.

Common choices include:

- ▶ Fourier bases,
- ▶ B-splines,
- ▶ wavelets.

Why Functional Representations?

Smoothing the data to recover a functional representation offers several advantages:

- ▶ Noise reduction through smoothing,
- ▶ Dimension reduction via basis coefficients,
- ▶ Continuous evaluation over the entire domain,
- ▶ Natural computation of derivatives,
- ▶ Analysis of irregularly sampled data.

Basis Expansions

The representation

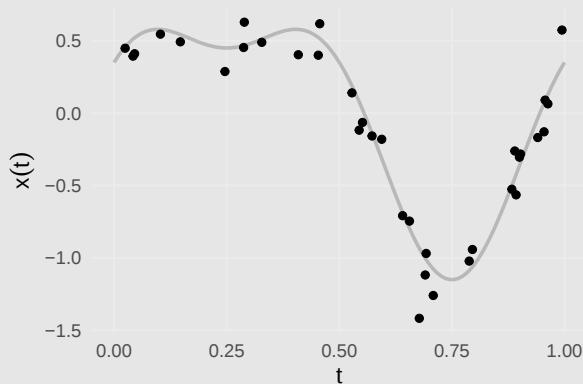
$$\tilde{x}(t) = \sum_{k=1}^K c_k B_k(t),$$

provides us with all of those advantages.

Learning such representations is also straightforward and boils down to minimizing a MSE criterion:

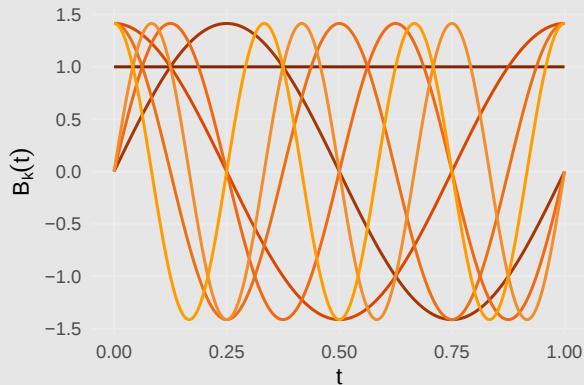
$$\sum_j \left[x(t_j) - \sum_{k=1}^K c_k B_k(t_j) \right]^2.$$

Example: Functional Smoothing



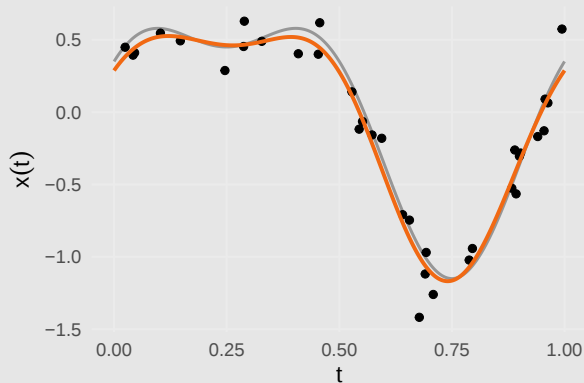
Underlying smooth process.

Example: Fourier Basis



Fourier basis functions.

Example: Smoothed Reconstruction



Recovered smooth functional representation.

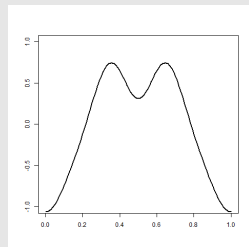
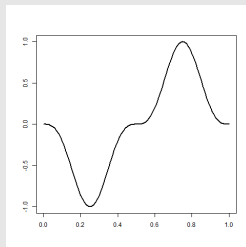
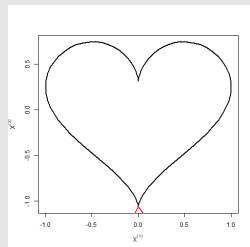
Functional representation of the contour

We use basis expansion to smooth the coordinate paths obtained with marching square.

It gives us a smooth, continuous and parametric representation of the contour.

We can use multivariate FDA approaches to solve statistical questions about the shapes.

Functional representation of the contour



Shapes and statistics

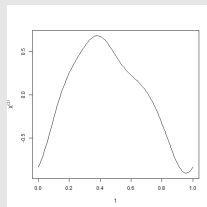
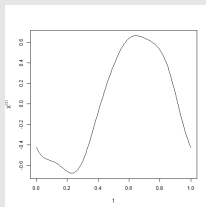
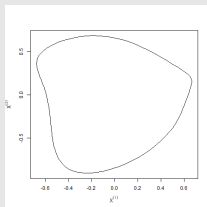
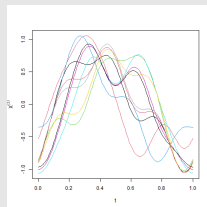
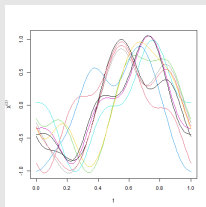
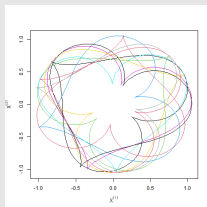
So what about repetition and data sets?

With repetition, different rotations and scales of the same object lead to completely different coordinate functions.

This starting point of the coordinate paths is arbitrary with respect to shape features.

Shapes and statistics

Contours need to be aligned first in order for the statistics to be meaningful.



Functional contour alignment

Functional contour alignment

In order to obtain a sample of shapes, we must first align the contour.

This is because shape is an geometric structure invariante with respect to some deformations.

These are:

- ▶ Translation
- ▶ Scale
- ▶ Rotation
- ▶ Path starting point (parameterization)

Our proposed model

The resulting contour we observed is parametrized as:

$$\mathbf{C}(t) = \rho \mathbf{O} \tilde{\mathbf{C}} \circ \gamma(t) + \mathbf{T} \quad (2)$$

where $(\rho, \mathbf{O}, \gamma, \mathbf{T})$ are the deformation parameters and $\tilde{\mathbf{C}}(t)$ the shape.

Functional contour alignment

The alignment procedure we propose estimates these deformations, allowing for their analysis.

It also allows the removal of their effects to analyze what remains: the shape.

Shape as Functional Data

We model closed curves as bivariate functional data.

Smoothing the data solves the problem of images and objects having different resolutions (irregular sampling).

Fourier bases are a natural choice given the cyclical nature of closed curves.

Representation using Fourier Basis

First, we represent contours using a Fourier basis.

$$\mathbf{C}(t) = \begin{pmatrix} X(t) \\ Y(t) \end{pmatrix} = \sum_{k=0}^M \begin{pmatrix} a_{k1} \\ a_{k2} \end{pmatrix} \psi_k(t), \quad t \in [0, 1], \quad (3)$$

with

$$\psi_0(t) = 1, \quad \psi_k(t) = \begin{cases} \sqrt{2} \sin((k+1)\pi t) & \text{if } k \text{ odd,} \\ \sqrt{2} \cos(k\pi t) & \text{if } k \text{ even,} \end{cases} \quad (4)$$

for $t \in [0, 1]$, $k = 1, \dots, M$, with M even.

Estimating the scale and translation

Estimating $\mathbf{T}_i = (T_x, T_y)$ and ρ_i is rather simple by assuming the shapes to be centered at $(0,0)$ and to have unitary norm.

$$\mathbf{T} = \int_0^1 \mathbf{C}(t) dt = \int_0^1 \sum_{k=0}^M \mathbf{a}_k \psi_k(t) dt = \mathbf{a}_0,$$
$$\rho = \|\mathbf{C} - \mathbf{T}\|_{\mathcal{H}} = \left\| \sum_{k=1}^M \mathbf{a}_k \psi_k \right\|_{\mathcal{H}} = \sqrt{\sum_{k=1}^M \mathbf{a}_{k2}^2},$$

This is extremely quick since it is given by the functional representation and can be done shape by shape independantly.

Estimating the scale and translation

After we estimate the translation and scale deformation, we obtain $\mathbf{C}^*$, which we call the pre-shape:

$$\mathbf{C}^*(t) = \frac{1}{\rho}(\mathbf{C}(t) - \mathbf{T})$$

Estimating the reparametrization and the rotation

The biggest challenge when developing our proposed approach.

The effects of these deformations are entangled.

We need to estimate both at the same time.

Starting point (reparametrization)

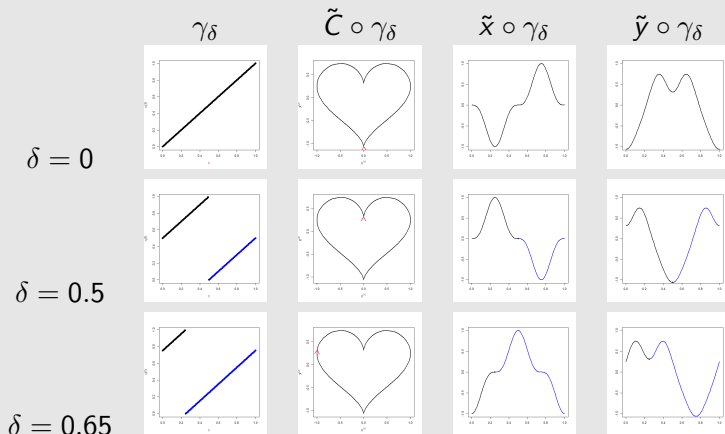
The reparametrization γ is a simple wrapping function that parametrize the effect of different starting point when traveling along the contour.

We define $\gamma \in \Gamma$, with

$$\Gamma = \{\gamma_\delta(t) = \text{mod}(t - \delta, 1), t \in [0, 1], \delta \in [0, 1]\}$$

Starting point (reparametrization)

we can visualise the effect of this function here:



Reparametrization

Because we analyse closed curves, the coordinate functions are cyclical.

The δ parameter dictates where on the contour did we begin travelling.

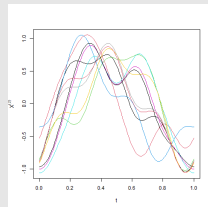
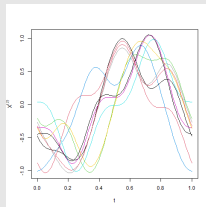
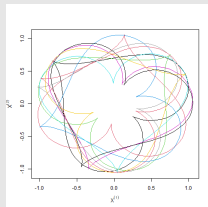
Reparametrization

One might think that we can wrap the functions until they are aligned.

But the coordinate functions are entirely different for different rotations.

These deformations must be estimated jointly.

Effect of the rotation on the coordinate functions



Rotation

The rotation $\mathbf{O}$ of the pre-shape is parametrized with a standard rotational matrix:

$$\mathbf{O} = \mathbf{O}_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

The problem boils down to estimating θ .

Estimating the reparametrization and the rotation

The first step to align the pre-shapes $\mathbf{C}^*$ is to define a template μ . The reparameterization or rotation do not really matter as long as they are the same for all pre-shapes.

The template can be:

- ▶ A random observation $\mathbf{C}_i^*$.
- ▶ Some version of the Karcher/Frechet mean.
- ▶ A specific observation aligned as desired.

Estimating the reparametrization and the rotation

Given a template $\boldsymbol{\mu}$, we seek to align the pre-shape $\mathbf{C}_i^*$ by finding the parameters δ and θ that aligns the best $\mathbf{C}_i^*$ to $\boldsymbol{\mu}$.

$$(\hat{\theta}, \hat{\delta}) = \arg \min_{(\theta, \delta) \in [0, 2\pi] \times [0, 1]} \|\mathbf{O}_\theta \mathbf{C}_i^* \circ \gamma_\delta - \boldsymbol{\mu}\|_{\mathcal{H}}^2. \quad (5)$$

Estimating the reparametrization and the rotation

There are no closed-form solutions.

However, we can find a solution for θ given a fixed δ ,

and, we can find a solution for δ given a fixed θ .

Estimating the reparametrization and the rotation

We can therefore solve the alignment issues by

- ▶ Searching on a grid (for δ) for an optimum value
- ▶ **Developping an iterative algorithm (ICF) that updates both parameters every other steps.**

After removing all of the deformation variables; we are left with the shape $\tilde{\mathbf{C}}$.

Estimating the reparametrization and the rotation

After removing all of the deformation variables; we are left with the shape $\tilde{\mathbf{C}}$.

$$\mathbf{C}^*(t) = \frac{1}{\rho}(\mathbf{C}(t) - \mathbf{T})$$
$$\tilde{\mathbf{C}}(t) = \mathbf{O}_\theta \mathbf{C}^*(t) \circ \gamma_{(\delta)}$$

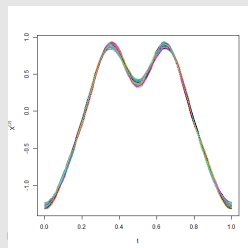
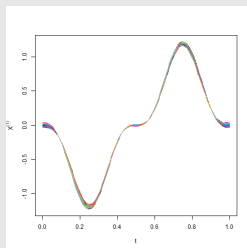
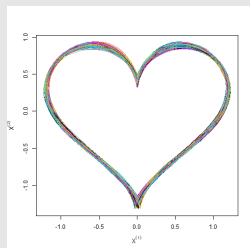
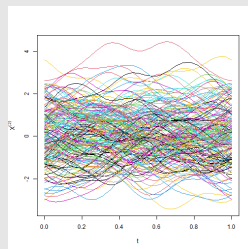
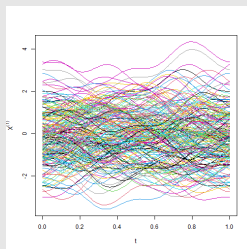
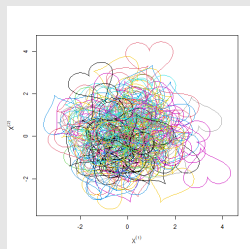
Alignement results

Before we go over the statistical analysis we conduct on shapes; let us make sure the alignement procedure works.

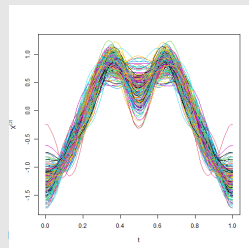
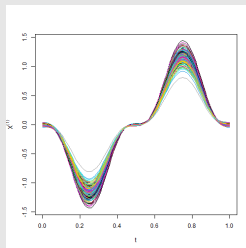
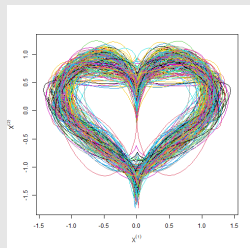
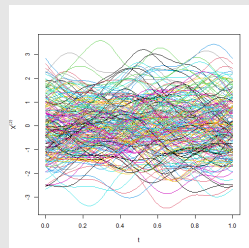
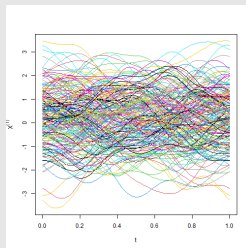
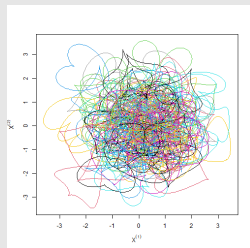
We deformed shapes and realigned them (simulations).

We also aligned real data.

Alignment on simulated data



Alignment on simulated data



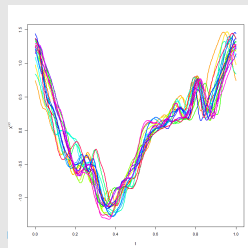
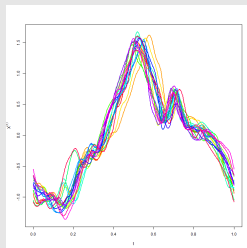
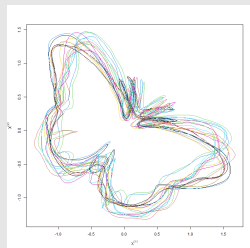
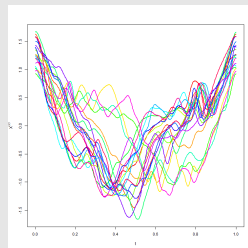
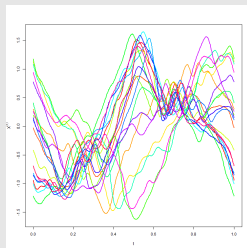
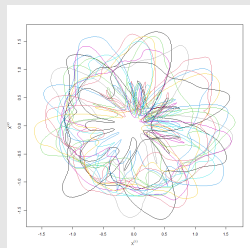
Aligment on real data

MPEG-7 database:



Figure: Examples of images from the database for the butterfly and fork objects.

Alignment on real data



Shape analysis : applications

Functional shape analysis

- ▶ So far, we have not done any *statistics*.
- ▶ We needed to prepare the data so that the statistical analysis is meaningful.
- ▶ At this point, we can consider multiple statistical problems related to shape and analyze both deformation variables (scalar) and shapes (functional) jointly.

Modeling $\mathbf{X}$ through a joint PCA approach

We propose a Functional Principal Component Analysis (FPCA) approach.

We extract features that can be used for both unsupervised and supervised learning problems.

Functional Principal component

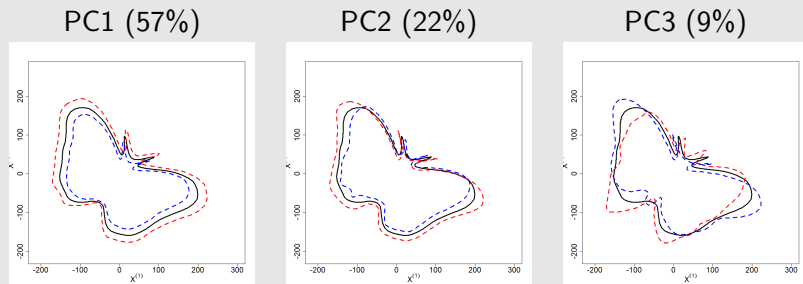


Figure: Plots of the estimated mean function $\bar{\mathbf{z}} = \sum_i \mathbf{z}_{i1}$ in black, of $\bar{\mathbf{z}} - 20\hat{\phi}_k$ in blue and of $\bar{\mathbf{z}} + 20\hat{\phi}_k$ in red, for $k = 1$ (first column), $k = 2$ (second column) and $k = 3$ (third column).

Generation

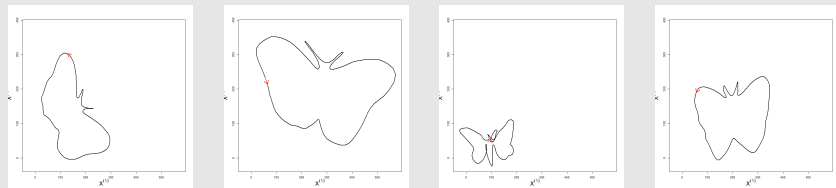


Figure: Butterfly curves generated with our approach with the deformation parameters.

Classification: Melanoma detection based on moles shape

- ▶ HAM10000 dataset contains photos of moles.
- ▶ We trained a model on 8000 images with two labels: melanoma or benign.
- ▶ Our goal was to compare the performance of a shape-based model against a pixel-based one.

Classification: Melanoma detection based on mole shapes

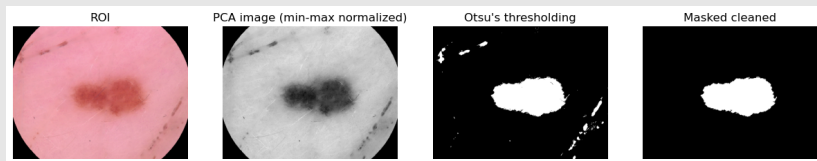


Figure: Segmentation pipeline.

Classification: Melanoma detection based on mole shapes

Model	avg-AUC	avg-Bal.Accuracy	avg-F1
Shape	0.640 ± 0.031	0.572 ± 0.036	0.837 ± 0.020
Pixel	0.697 ± 0.069	0.544 ± 0.049	0.814 ± 0.016

Table: Stratified-Nested-CV Results overview.

Classification: Melanoma detection based on mole shapes

Model	VRAM (GB)	avg-Training time (s)
Shape	0.7	31.7 ± 6.92
Pixel	66.7	174 ± 82.9

Table: Stratified-Nested-CV Results overview.

Extension to multiple contours

Multiple shapes

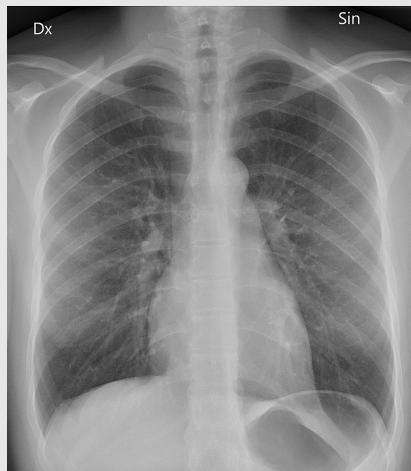
In images, an object of interest could be described by multiple contours interacting,

or the interest might be in considering multiple objects simultaneously.

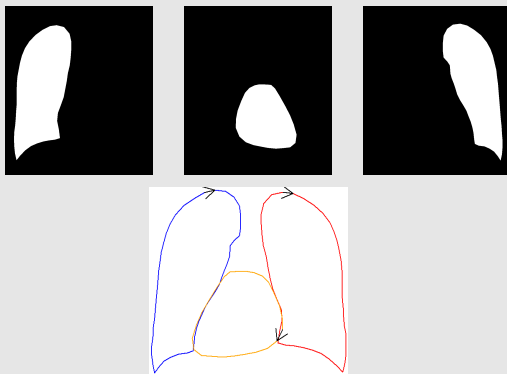
To consider multiple contours at once, a model needs to be specified.

Multiple shapes: motivating example

Chest X-rays used to identify patients with cardiomegaly.



Multiple shapes: motivating example



We define a **multivariate planar curve**

$\mathbf{C} = \text{Vec}(C_1, \dots, C_p) = (C_1^\top, \dots, C_p^\top)^\top$, where

$C_j = (X_j, Y_j)^\top \in \mathcal{H}$ are univariate curves.

The univariate model

$$\mathbf{C}(t) = \rho \mathbf{O} \tilde{\mathbf{C}} \circ \gamma(t) + \mathbf{T} \quad (6)$$

where $(\rho, \mathbf{O}, \gamma, \mathbf{T})$ are the deformation parameters and $\tilde{\mathbf{C}}(t)$ the shape.

Our proposed model for multiple contours.

Since all contours originate from the same image, translation, rotation, and scaling are considered global deformations.

$$C_j = \rho \mathbf{O}_\theta \tilde{C}_j \circ \gamma_{\delta_j} + \mathbf{T}, \quad j = 1, \dots, p, \quad (7)$$

Relative variations between contours are encoded directly in the multivariate shape variable $\tilde{\mathbf{C}} = \text{Vec}(\tilde{C}_1, \dots, \tilde{C}_p)$.

Naive model

A straightforward extension of the univariate model would look like:

$$C_j = \rho_j \mathbf{O}_{\theta_j} \tilde{C}_j \circ \gamma_{\delta_j} + \mathbf{T}_j, \quad j = 1, \dots, p, \quad (8)$$

but this implicitly removes the relative position, orientation, and scale of the objects.

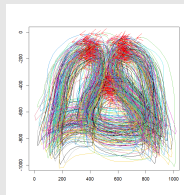
Multiple shapes

We developed methodologies for fundamental tasks of statistical shape analysis for the multivariate problem.

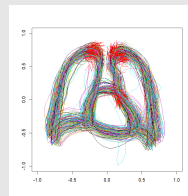
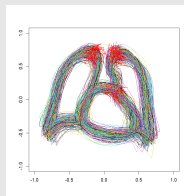
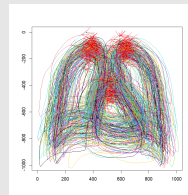
1. Alignement
2. Measure of dissimilarity
3. Estimating a mean shape
4. Classification



$Y = 1$



$Y = 0$



Multiple shapes: classification results

Classification accuracy with linear functional model:

Scenario	Method	GL1	GL2	PLS	PCR
Registered	<i>Multivariate</i>	79.65	77.34	85.22	82.94
	<i>Naive</i>	71.73	71.94	78.83	73.36
Added noise	<i>Multivariate</i>	80.50	78.39	85.99	82.77
	<i>Naive</i>	77.37	74.11	77.92	71.67

Table: Classification accuracies (%) obtained by 10-fold cross-validation.

Conclusion

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Statistical shape analysis can provide new information about images usefull in unsupervised and supervised analysis.

On their own, shapes can provide interpretable information lost in pixel-based approaches.

In order to analyze the underlying shapes of objects, we developed an alignment procedure.

The alignement procedure, the computation of the mean shape and statisticals analysis models were extended to considered multiple interacting contours jointly.

Current work

We are developping a surface representation for the colors inside objects of interest.

We are developping statistical test for color differences based on those surfaces.

We are also working on a vast R package to cover shape analysis end-to-end.

I would love to answer your questions.

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Alignement on real data: deformations

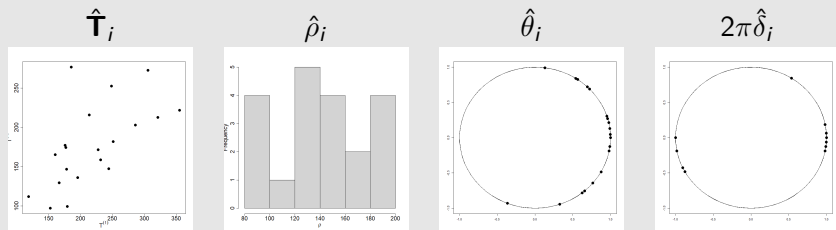


Figure: Plots of the estimated deformation parameters of each curve in both datasets

Breaking down the procedure

- ▶ Edge/contour detection for a collection of images.
- ▶ Extract an ordered list of pixels.
- ▶ Learn a functional representation for both coordinates.
- ▶ Estimate deformation parameters.
- ▶ Remove the deformations to obtain the shape as two univariate functions.
- ▶ Statistical analysis of the deformation variables and shapes.

Functional Data

In FDA, one observation $x_i(t)$ is therefore a function.

In FDA, a dataset consists of a collection of functions

$$S = \{x_i(t) : i = 1, \dots, n\},$$

where all observations are defined on a common domain T .

In practice, the underlying function cannot be observed continuously and is **only recorded at a finite number of sampling locations**.

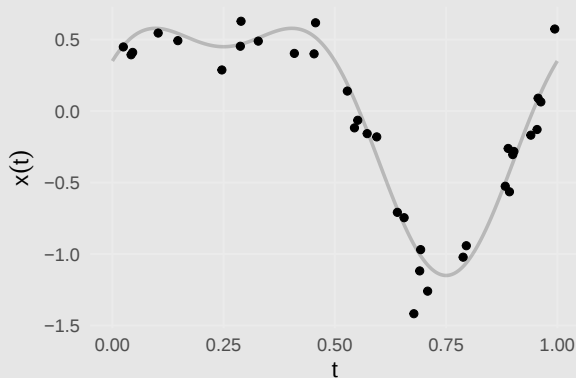


Figure: Discrete observations from an underlying smooth functional process.

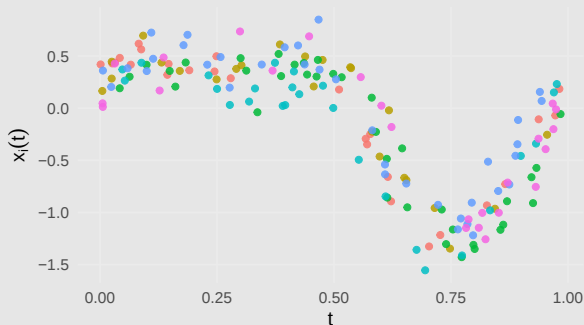


Figure: A sample of functional observations.

Note: The data may be discretely observed on different grids and sampled at a different number of locations.

Smoothing and Functional Representation

FDA commonly assumes that the observed data arise from an underlying smooth process contaminated by measurement error.

Observed measurements are therefore often smoothed before analysis.